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THE QFT SCATTERING RESONANCES CANNOT BE ASSOCIATED WITH THE VON NEUMANN — WIGNER BOUND STATES IN A CONTINUOUS SPECTRUM

L.G.Zastavenko

This work is our response to recent attempts to connect the observed resonances in, e.g., an electron-positron system, with the von Neumann — Wigner bound states in a continuous spectrum. Such states can exist only if the potential is not absolutely integrable. We study the Schroedinger equation $(-\Delta + V(r) - k^2)\psi = 0$, where the potential $V(r)$ is not absolutely integrable: $\int_0^\infty |V(r)|dr = \infty$ and the Fourier image $\tilde{V}(|\vec{q}|) = \int e^{i\vec{q}\vec{x}}V(r)d^3x$ has singularities for real values of $|\vec{q}|$, $|\vec{q}| = |\vec{q}_i| > 0$. We consider the perturbation theory expansion $A(\vec{p}, \vec{p}_0) = A_1 + A_2 + \ldots$ of the scattering amplitude. Our (trivial) result is that singularities of the function $\tilde{V}(q)$ give rise to singularities of quantities $A_1, A_2, \ldots$. But the QFT (quasi)potentials do not give any singula-rities of the Born amplitude A_1. Thus, our statement is that the computed by Arbuzov et al. and Spence and Vary resonances cannot be connected with the von Neumann — Wigner bound states.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Уровни фон Неймана — Вигнера и резонансные состояния систем электрон-позитрон, протон-протон

Л.Г.Заставенко

Работа является откликом на недавние попытки связать резонансы в системе электрон-позитрон со связанными состояниями в непрерывном спектре (фон Нейман — Вигнер). Такие состояния существуют только, если $\int_0^\infty |V(r)|dr = \infty$. Тогда фурье-образ $\tilde{V}(|\vec{q}|) = \int d^3xV(r)e^{i\vec{q}\vec{x}}$ имеет сингулярности при $|\vec{q}| > 0$. Мы рассматриваем задачу рассеяния для уравнений Шредингера. Пусть ряд теории возмущений $A(\vec{p}, \vec{p}_0) = A_1(\vec{p}, \vec{p}_0) + A_2(\vec{p}, \vec{p}_0) + \ldots$ задает амплитуду рассеяния. При нашем потенциале величины $A_1, A_2, \ldots$ — сингулярны. Но квазипотенциалы КТП дают несингулярные величины $A_1, A_2, \ldots$. Поэтому я думаю, что вычисленные Арбузовым и др. и Спенсом и Вэри резонансы не связаны с решениями фон Неймана — Вигнера.

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I. Introduction

Since the work[1] it became almost common knowledge that the Schroedinger equation

$$(-\frac{d^2}{dr^2} + V(r) - k^2)\psi = 0 \qquad (1.1)$$

with the potential

$$V(r) = \frac{2A}{r}\sin(2pr) \qquad (1.2)$$

at $k^2 = p^2$ has the solution which vanishes as r tends to infinity:

$$\psi(r) \to 0 \text{ as } r \to \infty. \qquad (1.3)$$

Introducing into the potential some additional degree of freedom, e.g., taking $V(r) = (2A/r)\sin(2pr + \delta)$, one can ensure fulfillment of the boundary condition

$$\psi(0) = 0. \qquad (1.4)$$

We shall call solutions of the Schroedinger equation which satisfy both boundary conditions (1.3) and (1.4), the bound states in a continuous spectrum[1]. (Note that two independent solutions of eq.(1.1), (1.2) at $k^2 = p^2$ can be represented as

$$\psi_{\pm}(r) = \sum_{n=0}^{\infty} \sum_{s=-\infty}^{+\infty} d_{sn} e^{ipr(2s+1)} r^{\pm\Lambda - n},$$

where $\Lambda = A/(2p)$ and D_{sn} are numerical coefficients to be determined from eq.(1.1)).

Recently there were observed some unexpected resonances in a system of charged particles[2-6].

Some authors tried to interpret these resonances in terms of the von Neumann — Winger bound states in a continuous spectrum[7,8]. These theoretical works are mainly computational. Spence and Vary, in particular, considered the scattering phase $\delta(k)$ of the electron-position system.

Space and Vary insist that their computations give

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} [\delta(k_0 + \varepsilon) - \delta(k_0 - \varepsilon)] = \pi \qquad (1.5)$$

for some values of momentum k_0.

1. This result is the starting point of our consideration. We state:

1) If the Fourier image of the potential $\widetilde{V}(|\vec{q}|) = \int e^{i\vec{q}\vec{x}} V(r) d^3x$ is singular at some point $|\vec{q}| = 2p$ and $\int_0^\infty |V(r)| dr = \infty$, $|V(r)| \le M < < \infty$, then the Schroedinger equation (1.1) has, at $k^2 = p^2$, a solution ψ which does satisfy boundary condition (1.3).

2) For such potential the first order perturbation theory scattering amplitude is singular at the point $k^2 = p^2$.

3) The QFT quasipotentials of works[7] and[8] do not give any singularities of the scattering amplitude in lowest orders of the perturbation theory. Thus, we think that attempts to connect the computed resonances[7,8] with the von Neumann — Winger bound states in a continuous spectrum are hopeless.

1.1. We will give also the following verbal description of the von Neumann — Winger phenomenon of bound states in a continuos sprectrum. At $k \approx p$ there arise a resonance of oscillations with frequencies $\pm k$ and $2p$ in our Schroedinger equation (1.1), (1.2): The oscillations with frequency $2p$ induce transition between the states which possess frequencies $\pm k$ (see also Appendix). This resonance does not allow a particle, if it is located somewhere in the region $r \sim 1$, to penetrate into the region $r >> 1$. And vice versa, if a particle is located in the region $r >> 1$, it cannot ponetrate into the region $r \sim 1$. Our potential (1.2), however small the value of the parameter A is, forms the resonance barrier, which is the more difficult to ponetrate, the smaller is the quantity $|k^2 - p^2|$ and which a particle cannot penetrate into if $k^2 = p^2$.

1.2. The potential (1.2) gives discontinuity in the energy dependence of the S-wave scattering phase (see eq.(2.7)). Let us construct the Jost solution $\psi_k(r)$ for the potential (1.2) (see eq.(A.1)). The point $k^2 = p^2$ is a branching point for this solution (see Appendix A).

1.3. Appendix B contains consideration of singularities of the amplitude $A_1(\vec{p}_1 \vec{p}_0)$.

II. Scattering on a Singular Potential

1. Let us consider the scattering problem for a Schroedinger equation in the momentum space:

$$(p^2 - k^2)\Psi(p) + \int V(\vec{p}, \vec{l})\Psi(\vec{l})d^3l = 0, \tag{2.1}$$

$$\psi(\vec{p}) = \delta(\vec{p} - \vec{p}_0) + \frac{A(\vec{p}, \vec{p}_0)}{p^2 - k^2 - i\varepsilon}, \quad \vec{p}_0^{\,2} = k^2, \tag{2.2}$$

$$\begin{aligned} A(\vec{p}, \vec{p}_0) &= - V(\vec{p}, \vec{p}_0) - \int V(\vec{p}, \vec{l}) \frac{d^3l}{l^2 - k^2 - i\varepsilon} V(\vec{l}, \vec{p}_0) - \ldots = \\ &= A_1(\vec{p}, \vec{p}_0) + A_2(\vec{p}, \vec{p}_0) + \ldots . \end{aligned} \tag{2.3}$$

Here $A(\vec{p}, \vec{p}_0)$ is the scattering amplitude.

2. Let us take at first a local potential

$$V(\vec{p}, \vec{l}) = \tilde{V}(\vec{p} - \vec{l}). \tag{2.4}$$

2.1. Even if the potential $V(r)$ in equation (1.1) is absolutely integrable, its Fourier image $\tilde{V}(|\vec{q}|)$ as a function of the variable $|\vec{q}| \equiv q$ can be nonanalytical at some points $q = q_i$, $|\mathrm{Im}\, q_i| \geq 0$ (see Appendix B). Suppose, however, that this function is analytical along all the axis $q \geq 0$. Then, the scattering amplitude (2.3) is also an analytical function of the momenta $\vec{p}, \vec{p}_0$, $|\vec{p}| = |\vec{p}_0| = k$, regular and bounded for real values of these momenta.

2.1.1. The potential $\tilde{V}(q)$ may have singularities in the complex plane, $q = q_i$, $i = 1, 2, 3, \ldots$ $\mathrm{Im}\, q_i \neq 0$. Even if the potential is real, these singularities are capable to cause characteristic peculiarities of parts A_1, A_2 and so on of the scattering amplitude. Let us take, e.g.,

$$\tilde{V}(q) = \frac{4\pi A}{q} \frac{\varepsilon}{\varepsilon^2 + (q - 2p)^2}, \tag{2.5}$$

where $|\varepsilon/p| << 1$.

One has

$$\tilde{V}(q) \to \frac{4\pi^2 A}{2p} \delta(q - 2p) \tag{2.6}$$

as $\varepsilon \to 0$. Such a characteristic behaviour of the quantity $\tilde{V}(q)$ leads to observable consequences.

2.2. The absolutely nonintegrable potential (1.2) corresponds to the limit $\varepsilon \to 0$ in eq.(2.6). For this potential the Born approximation gives no scattering at all if $|\vec{p}| = |\vec{p}_0| = k < p$ and gives scattering only on the cone $|\vec{p} - \vec{p}_0| = 2p$ if $|\vec{p}| = |\vec{p}_0| = k > p$. The Born approximation s-wave scattering phase for the potential (1.2) is:

$$\delta(k) = \begin{cases} 0, & k^2 < p^2 \\ \pi A, & k^2 > p^2. \end{cases} \qquad (2.7)$$

Thus, the potential (1.2) gives the rupture of the scattering phase.

In general, if the quantity $\tilde{V}(q)$ has a singularity at $q = 2p$, then the Born approximation s-wave scattering phase has a singularity at $k = p$.

2.3. Let us now discuss the quasipotential equation of the work by Arbuzov et al.[7]. It can be written down essentially as

$$[2\omega(\vec{p}) - 2\omega(k)]\psi(p) + \lambda \int V(\vec{p}, \vec{l})\psi(\vec{l})d^3l = , \qquad (2.8)$$

$$V(\vec{p}, \vec{l}) = [\omega(\vec{p})\omega(\vec{l})\, |\vec{p} - \vec{l}\,|\,]^{-1}\, P[\omega(\vec{p}) + \omega(\vec{l}) + |\vec{p} - \vec{l}\,| - 2\omega(k)]^{-1},$$

here the potential does depend on the center of mass energy $2\omega(k)$ of the system $\omega(k) = (m^2 + k^2)^{1/2}$, symbol P denotes a principal value (in the work[7] the authors use complex denominator $\omega(\vec{p}) + \omega(\vec{l}) + |\vec{p} - \vec{l}\,| - 2\omega(k) - i\varepsilon$. This prescription seems to us to be obviously wrong, for it leads to the energy nonconservation in the process of scattering, see, e.g., eq. (2.3). This remark does not influence the computational results of work[7] as they use the approximation neglecting the imaginary part of the potential). Then, one has

$$A_1(\vec{p}, \vec{p}_0) = \lambda\omega(k)^{-2}|\vec{p} - \vec{p}_0|^{-2}. \qquad (2.9)$$

Here $|\vec{p}\,| = |\vec{p}_0| = k$ is the energy conservation law. Equation (2.9) exhibits no singularities in its dependence on the parameter k along the real axis. The quantity (2.9) is not capable of describing any resonances of the scattering cross-section. Let us now consider the s-wave part of the quantity $A_2(\vec{p}, \vec{p}_0)$. It is easy to prove that this function also has no singularities on the real k axis.

2.3.1. Thus, we have to state about the results of the works[7] and[8]: these results have nothing to do with the von Neumann and Wiegner bound states in a continuous spectrum. The latter cause singularities of the scattering amplitude which can be observed, e.g., in the Born approximation whereas Arbusov et al. claim their levels to be essentially nonperturbative.

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Appendix A

Jost Solution of Eq.(1.1), (1.2)

Let us denote $\psi_k(r)$ the Jost solution of eq.(1.1), (1.2), as determined by the condition

$$\psi_k(r) \sim e^{ikr} \text{ as } r \to \infty. \tag{A.1}$$

One has

$$\psi_k(r) = \sum_{s=-\infty}^{+\infty} e^{i(k+2ps)r} B_s(r), \tag{A.2}$$

$$2ps\,2(k+ps)B_s - 2i(k+2ps)\frac{dB_s}{dr} - \frac{d^2B_s}{dr^2} = iA(B_{s-1} - B_{s+1})/r. \tag{A.3}$$

It follows from eq.(A.1) that

$$\begin{aligned} &B_s(r) \to 0 \text{ as } r \to \infty,\ s \neq 0 \\ &B_0(r) \to 1 \text{ as } r \to \infty. \end{aligned} \tag{A.4}$$

When $|\varepsilon| << |p|$, $\varepsilon = k - p$, equations (A.3) reduce to the system of two coupled equations for functions $B_0(r)$ and $B_{-1}(r)$. One gets

$$B_0(r) \approx \sum_{m=0}^{\infty} \frac{\Gamma(m+\Lambda)\Gamma(m-\Lambda)}{m!\Gamma(\Lambda)\Gamma(-\Lambda)\,(2i\varepsilon r)^m}. \tag{A.5}$$

Barnes integral representation[9] enables one to prove formula

$$B_0(r) \approx \alpha(\Lambda)\,(2i\varepsilon r)^{\Lambda} + \alpha(-\Lambda)\,(2i\varepsilon r)^{-\Lambda} \tag{A.6}$$

if $1 << r << \frac{1}{|\varepsilon|}$. Standard sewing procedure gives representation

$$\psi_k(r) \approx a(\Lambda, \varepsilon)\psi_+(r) + a(-\Lambda, \varepsilon)\psi_-(r) \tag{A.7}$$

of the Jost solution if $|\varepsilon| r << 1$. One has $a(\Lambda, \varepsilon) \sim \varepsilon^{\Lambda}$: it looks like the point $k^2 = p^2$ is the branching point of the Jost solution.

Appendix B

Here we shall point out that if potential $V(r)$ contains part $\delta V(r) = \sim \sin(2pr + \delta) f_a(r)$, where $f_a(r) \sim r^{-a}$ as $r \to \infty$, its Fourier image $\tilde{V}(q)$ has singularity at $q = 2p$, and the Born approximation S-phase $\delta(k)$ has singularity at $k = p$.

Note Added in Proof

The QFT quasipotential approach gives the s-wave Schroedinger equation of the type

$$(-\frac{d^2}{dr^2} - k^2)\psi(r) + \int V(r,r')\psi(r')dr' = 0, \tag{N.1}$$

where $V(rr') \cong O(\exp(-m\sqrt{r^2 + r'^2}))$ as $r \to \infty$. Let us consider the Schroedinger equation (N.1) with saparable potential $V(r, r) = \lambda\, e^{-(r + r')}$. The corresponding eq. (N.1) for $k^2 = k_0^2$ has solution $\psi_0(r) = e^{-r}$; here the value is determined by the equation

$$(1 + k_0^2) = \lambda \int_0^\infty e^{-2r'} dr' . \tag{N.2}$$

Of course, our solution $\psi_0(r)$ does not satisfy condition (1.4). Nevertheless, this example shows, in principle, that the Schroedinger equation (N.1) with nonlocal potential is capable to have bound states in continuons spectrum (i.e. it can have for some values of k^2, $k^2 > 0$, solutions, which satisfy both boundary conditions (1.3) and (1.4)). This general fact explains calculations of works[7] and[8]. These AShBS bound states resemble, to some extent, the von Neuman — Wigner bound states, but, unlike these states, their descent is connected with the non-locality of the potential.

Let us consider the solution $\psi(r, k)$ of our equation (N.1) which is determined by the boundary equation (1.4) and by the condition $\frac{d}{dr}\psi(r, k)\,|_{r=0} = 1$. One has $\psi(r, k) = |A(k)| \sin(kr + \delta(k))$ as $r \to \infty$. One has also

$$A(k_0) = 0 \tag{N.3}$$

if $\psi(r, k_0) \to 0$ as $r \to \infty$. Equation (N.3) ensures fulfillment of the condition (1.5) and thus explains the Spence — Vary phenomenon of the scattering phase discontinuity at the energy of the bound state in continuous spectrum

Note also, that eq. (N.2) shows that no bound state (in continuous spectrum) exist if $\lambda \to 0$. This observation explains the nonperturbativ character of the AShBS bound states[7,8].

The expression

$$f(\theta) = \sum_{l=0}^{\infty} \frac{e^{2i\delta_l(k)} - 1}{2ik} p_l(\cos\theta)$$

of the scattering amplitude shows that the Spence — Vary π discontinuity of the scattering phase cannot be observed. Now, in order to accept or reject the AShBS conjecture conserning the connection between the observed resonanses and bound states in continuous spectrum, one has to investigate the analytical behaviour of the function $\delta(k)$ in a vicinity of the point $k^2 = k_0^2$. If this function is analytical (except for π discontinuity at the point $k^2 = k_0^2$), the AShBS conjecture is probably wrong.

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